# Mean-field theory

Consider a d-dimensional Ising model with FM interactions in zero external magnetic field

$$H = -J\sum_{(ij)} s_i s_j$$

(ij – nearest neighbours)
(summation over links of neighbours)

Mean field (MF) assumes that the gegrees of freedom weakly fluctuate on top of their average values

$$s_i = m + \underbrace{(s_i - m)}_{\text{Treat as a small operator}}$$

$$H = -J\sum_{(ij)} [m + (s_i - m)][m + (s_j - m)] =$$

$$= -J\sum_{(ij)} [m^2 + 2m(s_i - m)] - J\sum_{(ij)} \cancel{(s_i - m)(s_j - m)}$$

Neglect

## The MF Hamiltonian

$$H_{MF} = J\sum_{(ij)} (m^2 - m s_i - m s_j)$$

Summation over pairs of nearest neighbours

Switching to the summation over sites instead

$$H_{MF} = \sum_i \left(\tfrac{1}{2} J z m^2 - J z m s_i\right)$$

The Ham-n is thus decomposed into those of independent spins — that may be solved exactly

\# of sites in the system

of independent spins

exactly

$$Z = (Z_{1spin})^N = \left[e^{-\frac{\beta J z m^2}{2}} \underbrace{\left(e^{m\beta J z} + e^{-m\beta J z}\right)}_{2\cosh(m\beta J z)}\right]^N$$

$N$ — # of sites in the system

The free energy

$$F(m) = -NT \ln\left[2e^{-\frac{\beta J z m^2}{2}} \cosh(m\beta J z)\right]$$

$$\ln(\cosh(x)) = \frac{x^2}{2} - \frac{x^4}{12} + \dots$$

Expansion in small $m$:

$$\boxed{\frac{F(m)}{N} = m^2\left(\frac{1}{2}zJ - \frac{1}{2}\frac{J^2 z^2}{T}\right) + \frac{1}{12} m^4 \frac{J^4 z^4}{T^3} + \dots}$$

(m - independent terms dropped)

- Ginzburg - Landau functional

A transition at $T_c = Jz$ !

$T > T_c$ - paramagnet $(m = 0)$

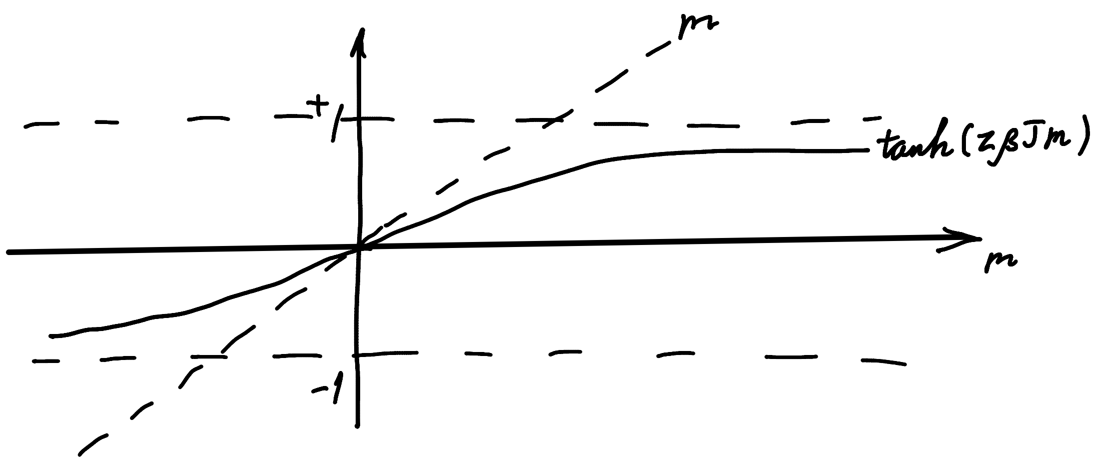
$T < T_c$ - ferromagnet $(m \neq 0)$

Self - consistency equation

$$m = \langle S_i \rangle_{MF}$$

$$\langle S_i \rangle = \frac{\mathrm{Tr}(e^{-\beta H_{MF}} S_i)}{\mathrm{Tr}(e^{-\beta H_{MF}})} = \tanh(z\beta J m)$$

$$\boxed{m = \tanh(z\beta J m)}$$




when $z\beta J > 1$ there exists a non-trivial solution $(m \neq 0)$, which leads to the critical temperature $T_c = zJ$

Equivalence between the self-consistency equation and the minimisation of the free energy:

$$m = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H_{MF}} s_i\right) \longleftrightarrow \frac{\partial F}{\partial m} = 0$$

---

## Limits of applicability

For $z \sim 1$ (# number) (like on most lattices, such as triangular and square lattices), there are no small parameters in the system $\rightarrow$ MF is uncontrolled

„Perturbation" $H - H_{MF} = -J \sum_{\langle ij \rangle} (s_i - m)(s_j - m)$

Thermodynamic perturbation theory

$$-\frac{F}{T} \quad T\left(e^{-\frac{H_0 + V}{T}}\right) = \mathrm{Tr}\left(e^{-\frac{H_0}{T}}\left(1 - \frac{V}{T} + \frac{V^2}{2T^2} + \dots\right)\right)$$

$$e^{-\frac{F}{T}} = Tr\left(e^{-\frac{H_0+V}{T}}\right) = Tr\left(e^{-\frac{H_0}{T}}\left(1-\frac{V}{T}+\frac{V^2}{2T^2}+\dots\right)\right)$$

Taking the log and expanding,

$$\boxed{F = F_0 + \langle V\rangle_0 - \frac{1}{2T}\left\langle (V-\langle V\rangle_0)^2\right\rangle_0 + \dots}$$

$$\langle H - H_{MF}\rangle_{MF} = -J\left\langle \sum_{(ij)} (S_i-m)(S_j-m)\right\rangle_{MF} =$$

$$= -J\sum_{(ij)} \langle S_i - m\rangle_{MF}\,\langle S_j - m\rangle_{MF} = 0$$

The 2nd order

$$-\frac{1}{2T}\langle (H-H_{MF})^2\rangle_{MF} = -\frac{1}{2T}\sum_{(ij)}\left\langle (S_i-m)^2\right\rangle_{MF}\left\langle (S_j-m)^2\right\rangle_{MF} J^2 \lesssim$$

$$\lesssim N z m^4 \frac{J^2}{T} \qquad (1)$$

For $T \sim zJ$, $(1) \sim N m^4 J$

The 4-th-order term in the MF Ginzburg-Landau functional $F^{(4)} \sim N z m^4 \frac{J^4 z^4}{T^3} \underset{T\sim T_c}{\simeq} \frac{N m^4 J}{z^2}$

---

Critical indices

$m^2 \propto T_c - T$

$\xi^2 \propto |T - T_c|$

$\xi \propto \tau^{-\nu}$ ; $\nu = \frac{1}{2}$ - MF critical index

$\chi \propto \tau^{-\gamma}$ ; $\gamma = 1$

$c \propto \tau^{-\alpha}$ ; $\alpha = 0$